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# Aharonov–Bohm Shift as a Shift in Normal Coordinates

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The Aharonov–Bohm shift in a closed system is considered. The solenoid is a charged, rotating cylinder which is electrically neutral. This model of Henneberger and Opatrny has a Hamiltonian which is a quadratic form. This quadratic form is transformed to normal coordinates, so that the stationary states become self-evident. It is shown that, in the original system, it is the kinetic angular momentum which is quantized. Solutions of the problem for an electron inside the solenoid are discussed. It is shown that the rotating cylinder exhibits different behavior if the electron is in the magnetic field or if it is in the external region. An external field approximation which replaces the cylinder by a constant magnetic field therefore cannot yield a correct solution of the Schrödinger equation which is continuous at the surface of the solenoid.

## 1. INTRODUCTION

The Aharonov–Bohm (1959) effect is a shift in the interference pattern of an electron beam caused by the introduction of a whisker of magnetic flux. The electron beam must be split into two paths, one passing the flux whisker on each side.

It is often stated that this effect is an effect of the vector potential, a quantity that is not observable. This viewpoint, while not wrong, is misleading.

The Aharonov–Bohm (AB) effect has its roots in classical electrodynamics. It was already known by Thomson (1904) that, in Coulomb gauge, the quantity $(e/c)\mathbf{A}(\mathbf{r})$ is actually the electromagnetic momentum due to the electron's electric field and the magnetic field of the solenoid (flux whisker). This fact has received far too little attention in standard texts on electrodynamics. It does appear in the text by Konopinski (1981).

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While it is true that the electron experiences no force in the AB effect, it is not true that there is no transfer of momentum. There is a changing electromagnetic momentum, and its rate of change must be balanced by a force on the solenoid. This has been discussed by Al-Jaber and Henneberger (1992). Moreover, since the electron's magnetic field penetrates the solenoid, there must be an exchange of energy between the total magnetic field and the source of the current in the solenoid, as discussed by Zhu and Henneberger (1990).

These considerations led Henneberger and Opatrny (1994) (HO) to consider an isolated system consisting of an electron and a spinning charged cylinder. The cylinder has a compensating charge on its axis, so that the only field in the external region is magnetic. HO considered a classical electron interacting with a massive quantum cylinder. The HO result is that the cylinder's wave function acquires a phase shift which is equal in magnitude but opposite in sign to that of the usual electron wave function phase shift. An analysis similar to that of HO was carried out for the Aharonov–Casher (1984) effect by Yu and Henneberger (1996). These authors showed further that in any closed system, infinitesimal phase shifts must always add to zero. The derivation is almost parallel to things that can be found in textbooks, but the result is important. As a consequence of this theorem, one may conclude that such phase shifts cannot affect statistics. Thus, if anyons exist, they can have no connection with Lagrangian dynamics.

The HO result may not seem compelling to some readers. It is therefore worthwhile to consider solutions of the Schrödinger equation for the complete isolated system considered by HO.

In this work, the system considered by HO is altered slightly. The line charge along the axis of the cylinder is replaced by a stationary charged cylinder having radius $r = a + \epsilon$, where $\epsilon$ is infinitesimal. The cylinder carries a surface charge $-\sigma$. In this way, all electric fields have been eliminated (except for the infinitesimal region between the cylinders). The magnetic moment of inertia of the rotating cylinder is easily shown to be unchanged from the value given by HO.

The change described above allows a discussion of the electron–magnetic field interaction in the interior region of the solenoid, as well as the exterior region. The present treatment gives the first demonstration of the physical problems involved in the solution of the Schrödinger equation at the boundary of the rotating cylinder.

## 2. NORMAL COORDINATES IN THE MODIFIED HO MODEL FOR AN ELECTRON IN THE EXTERIOR REGION

The Lagrangian of the system is

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2 + \frac{e}{c}\frac{\lambda}{2\pi}\dot{\theta}\dot{\varphi} + \frac{1}{2}I\dot{\theta}^2 \tag{1}$$

where $\mu$ is the electron mass, $r$ and $\varphi$ are the electron coordinates, and $\theta$ is the angle turned by some fiducial mark on the rotating cylinder. The vector potential at a distance $r$ is

$$A_\varphi = \frac{\Phi}{2\pi r}, \qquad \text{with} \qquad \Phi = \lambda\dot{\theta} \tag{2}$$

The constant $\lambda$ is given by

$$\lambda = 4\pi^2 a^3 \sigma / c \tag{3}$$

where $\sigma$ is the charge/cm$^2$ on the rotating cylinder and $a$ is the radius of the cylinder (assumed to be very small). The moment of inertia of the cylinder is the total moment, i.e., the sum of the mechanical and electromagnetic moments of inertia. Details are given in HO.

The dynamic nature of the AB effect is already evident in equation (1). The interaction of the electron with the vector potential has been replaced by the interaction with its source, the rotating charged cylinder.

The most direct (as well as the most enlightening) method of treating the Lagrangian is to transform it to normal coordinates. We carry out an orthogonal transformation to new variables $\tilde{\theta}$, $\tilde{\varphi}$ such that

$$\begin{aligned} \tilde{\theta} &= \theta \cos \eta + \varphi \sin \eta \\ \tilde{\varphi} &= -\theta \sin \eta + \varphi \cos \eta \end{aligned} \tag{4}$$

The inverse transformation is, of course,

$$\begin{aligned} \theta &= \tilde{\theta} \cos \eta - \tilde{\varphi} \sin \eta \\ \varphi &= \tilde{\theta} \sin \eta + \tilde{\varphi} \cos \eta \end{aligned} \tag{5}$$

We assume the angle $\eta$ to be time-independent. This assumption will be seen to be justified in the limit $I >> \mu r^2$. If this inequality fails, $\eta$ will depend on $r$, which in turn depends on time.

This assumption yields

$$\begin{aligned} \dot{\theta} &= \dot{\tilde{\theta}} \cos \eta - \dot{\tilde{\varphi}} \sin \eta \\ \dot{\varphi} &= \dot{\tilde{\theta}} \sin \eta + \dot{\tilde{\varphi}} \cos \eta \end{aligned} \tag{6}$$

The Lagrangian of equation (1) becomes

$$L = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2(\dot{\tilde{\theta}}^2 \sin^2 \eta + \dot{\tilde{\varphi}}^2 \cos^2 \eta + 2\dot{\tilde{\theta}}\dot{\tilde{\varphi}} \sin \eta \cos \eta)$$

$$+ \frac{e}{c}\frac{\lambda}{2\pi}(\dot{\tilde{\theta}}^2 \sin \eta \cos \eta - \dot{\tilde{\varphi}}^2 \sin \eta \cos \eta + \dot{\tilde{\varphi}}\dot{\tilde{\theta}} \cos^2 \eta - \dot{\tilde{\varphi}}\dot{\tilde{\theta}} \sin^2 \eta)$$

$$+ \frac{1}{2} I(\dot{\tilde{\theta}}^2 \cos^2 \eta - 2\dot{\tilde{\theta}}\dot{\tilde{\varphi}} \sin \eta \cos \eta + \dot{\tilde{\varphi}}^2 \sin^2 \eta) \tag{7}$$

The angle $\eta$ is chosen to have a value that causes the $\dot{\tilde{\theta}}\dot{\tilde{\varphi}}$ term to vanish. This condition yields

$$\mu r^2 \sin \eta \cos \eta + \frac{e\lambda}{2\pi c} (\cos^2 \eta - \sin^2 \eta) - I \sin \eta \cos \eta = 0 \tag{8}$$

Equation (8) has the solution

$$\tan 2\eta = \frac{e\lambda}{\pi c(I - \mu r^2)} \simeq \frac{e\lambda}{\pi c I} \tag{9}$$

For $I >> \mu r^2$, the assumption $\dot{\eta} = 0$ is valid. Interest here is in the case

$$\sin \eta \simeq \frac{e\lambda}{2\pi c I} \qquad \text{and} \qquad \cos \eta \simeq 1 \tag{10}$$

The Lagrangian then becomes

$$L = \frac{1}{2} \mu \dot{r}^2 + \left[\frac{1}{2} \mu r^2 \sin^2 \eta + \frac{e\lambda}{2\pi c} \sin \eta \cos \eta + \frac{1}{2} I \cos^2 \eta\right] \dot{\tilde{\theta}}^2$$
$$+ \left[\frac{1}{2} \mu r^2 \cos^2 \eta - \frac{e\lambda}{2\pi c} \sin \eta \cos \eta + \frac{1}{2} I \sin^2 \eta\right] \dot{\tilde{\varphi}}^2 \tag{11}$$

The canonical momenta are

$$\wp_r = \frac{\partial L}{\partial \dot{r}} = \mu \dot{r}$$

$$\wp_{\tilde{\theta}} = \frac{\partial L}{\partial \dot{\tilde{\theta}}} = \left[\mu r^2 \sin^2 \eta + \frac{e\lambda}{\pi c} \sin \eta \cos \eta + I \cos^2 \eta\right] \dot{\tilde{\theta}} \tag{12}$$

$$\wp_{\tilde{\varphi}} = \frac{\partial L}{\partial \dot{\tilde{\varphi}}} = \left[\mu r^2 \cos^2 \eta - \frac{e\lambda}{\pi c} \sin \eta \cos \eta + I \sin^2 \eta\right] \dot{\tilde{\varphi}}$$

It is convenient to introduce new quantities $\tilde{I}$ and $\tilde{\mu}$:

$$\tilde{I} = \mu r^2 \sin^2 \eta + \frac{e\lambda}{\pi c} \sin \eta \cos \eta + I \cos^2 \eta$$
$$\tilde{\mu} r^2 = \mu r^2 \left[\cos^2 \eta - \frac{e\lambda}{\pi c} \frac{\sin \eta \cos \eta}{\mu r^2} + \frac{I}{\mu r^2} \sin^2 \eta\right] \tag{13}$$

In the limit $I \to \infty$, $\tilde{I} \to I$ and $\tilde{\mu} \to \mu$.

The Lagrangian now has the simple form

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\tilde{\mu}r^2\dot{\tilde{\varphi}}^2 + \frac{1}{2}\tilde{I}\dot{\tilde{\theta}}^2 \tag{14}$$

and the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\tilde{\mu}r^2\dot{\tilde{\varphi}}^2 + \frac{1}{2}\tilde{I}\dot{\tilde{\theta}}^2 \\ &= \frac{\wp_r^2}{2\mu} + \frac{\wp_{\tilde{\varphi}}^2}{2\tilde{\mu}r^2} + \frac{\wp_{\tilde{\theta}}^2}{2\tilde{I}} \end{aligned} \tag{15}$$

The purpose of this section is the discussion of phase shifts. These are first order in the electronic charge $e$. We note that $\tilde{I}$ differs from $I$ by a term of order $e^2/I$. The quantity $\tilde{\mu}$ differs from $\mu$ by a term of the same order. We may therefore put $\tilde{\mu} = \mu$ and $\tilde{I} = I$ in equation (15).

The Hamiltonian of equation (15) then becomes

$$H = \frac{\wp_r^2}{2\mu} + \frac{\wp_{\tilde{\varphi}}^2}{2\mu r^2} + \frac{\wp_{\tilde{\theta}}^2}{2I} \tag{16}$$

The eigenstates of this Hamiltonian can be immediately written down. They are

$$\psi(k, r, \tilde{\varphi}, \tilde{\theta}) = J_m(kr)e^{im\tilde{\varphi}}e^{iM\tilde{\theta}} \tag{17}$$

with $m$ and $M$ integers, and the electron kinetic energy given by $\hbar^2k^2/2\mu$. We must, of course, exclude the $m = 0$ states; these are nonvanishing at $r = 0$. The remaining states form a complete set on the space of functions that vanish at $r = 0$. We assume the radius of the cylinder to be effectively zero.

Equations (1) and (16) show $\wp_\theta$, $\wp_\varphi$, $\wp_{\tilde{\theta}}$, $\wp_{\tilde{\varphi}}$ to be conserved quantities. In order to understand the result of this work, we must identify the physical significance of the angles $\tilde{\theta}$ and $\tilde{\varphi}$. We have

$$\begin{aligned} \wp_\theta &= I\dot{\theta} + \frac{e\lambda}{2\pi c}\dot{\varphi}, \qquad & \wp_{\tilde{\theta}} &= I\dot{\tilde{\theta}} = M\hbar \\ \wp_\varphi &= \mu r^2\dot{\varphi} + \frac{e\lambda}{2\pi c}\dot{\theta} \qquad & \wp_{\tilde{\varphi}} &= \mu r^2\dot{\tilde{\varphi}} = m\hbar \end{aligned} \tag{18}$$

$$\wp_\varphi - \wp_{\tilde{\varphi}} = \mu r^2(\dot{\varphi} - \dot{\tilde{\varphi}}) + \frac{e\Phi}{2\pi c} \tag{19}$$

With the second of equations (6), this becomes

$$\wp_\varphi - \wp_{\tilde{\varphi}} = \mu r^2\dot{\tilde{\theta}}\sin\eta + \frac{e\Phi}{2\pi c} = \mu r^2\frac{e\Phi}{2\pi cI} + \frac{e\Phi}{2\pi c}$$

The approximation $\mu r^2 << I$ then gives

$$\wp_\varphi - \wp_{\tilde{\varphi}} = -\hbar\alpha \tag{20}$$

with $\alpha = -e\Phi/ch$, as defined by AB.

Equation (20) shows that $\wp_{\tilde{\varphi}}$ is the kinetic angular momentum of the electron. In the AB problem, it is the kinetic angular momentum (*not* the canonical angular momentum) which is quantized. This writer has been insisting on this for the past 16 years (Henneberger, 1981).

The angles $\tilde{\theta}$ and $\tilde{\varphi}$ are just the angles that $\theta$ and $\varphi$ would have in the absence of any interaction. Multiplication of the first of equations (5) by the (conserved) canonical angular momentum of the cylinder yields

$$M\hbar(\theta - \tilde{\theta}) = -\tilde{\varphi}M\hbar \sin\eta = -M\frac{e\lambda\hbar}{2\pi cI}\varphi \tag{21}$$

where terms of order $e^2$ have been neglected in the last equality. The relation $M\hbar\lambda/I = \Phi$ yields

$$M\hbar(\theta - \tilde{\theta}) = \frac{-e\Phi}{ch}\hbar\varphi = \alpha\hbar\varphi \tag{22}$$

Equation (22) may be written

$$\Delta\int \wp_\theta\, d\theta = -\Delta\int \wp_\varphi\, d\varphi \tag{23}$$

where the symbol $\Delta$ refers to the change in these quantities due to the AB interaction. Equation (23) is just the result of HO.

For the skeptical reader, we provide a second proof that it is the kinetic angular momentum that takes on integral values of $\hbar$ in the AB problem.

We combine the equations

$$\frac{\hbar}{i}\frac{\partial}{\partial\tilde{\varphi}}\psi = m\hbar\psi \tag{24}$$

and

$$\frac{\partial}{\partial\tilde{\varphi}} = \cos\eta\frac{\partial}{\partial\varphi} - \sin\eta\frac{\partial}{\partial\theta} \tag{25}$$

The approximation $\sin\eta \approx \tan\eta$ and $\cos\eta \simeq 1$ yields

$$m\hbar\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial\varphi} - \frac{e\lambda}{2\pi cI}\frac{\hbar}{i}\frac{\partial\psi}{\partial\theta} \tag{26}$$

Putting

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial\theta} = I\dot{\theta}\psi \qquad \text{and} \qquad \lambda\dot{\theta} = \Phi$$

then yields

$$m\psi = \frac{1}{i}\frac{\partial\psi}{\partial\varphi} - \frac{e\Phi}{ch}\psi \tag{27}$$

Equation (27) shows that when the original canonical angular momentum operator acts on an eigenstate of $(1/i)\,\partial/\partial\tilde{\varphi}$, we have

$$m = \frac{1}{\hbar}L_{\varphi\text{can}} - \frac{e\Phi}{ch} = \frac{1}{\hbar}L_{\varphi\text{can}} + \alpha \tag{28}$$

Again, we see that in the AB problem the eigenvalues of $(1/\hbar)L_{\varphi\text{can}}$ are $m - \alpha$, where $m$ is an integer.

## 3. THE INTERIOR REGION OF THE CYLINDER

In the interior region of the rotating cylinder, the problem has the Lagrangian

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2 + \frac{e\lambda r^2}{c\pi a^2}\dot{\theta}\dot{\varphi} + \frac{1}{2}I\dot{\theta}^2 \tag{29}$$

The canonical momenta are

$$\wp_\theta = \frac{\partial L}{\partial\dot{\theta}} = I\dot{\theta} + \frac{e\lambda r^2\dot{\varphi}}{c2\pi a^2} \tag{30}$$

$$\wp_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = \mu r^2\dot{\varphi} + \frac{e\lambda r^2\dot{\theta}}{c2\pi a^2}, \qquad \wp_r = \frac{\partial L}{\partial\dot{r}} = \mu\dot{r}$$

Because of the factor $r^2$ in the interaction term of equation (29), it is no longer possible to transform to normal coordinates by means of a single rotation, as in the previous section. Fortunately, a satisfactory discussion is possible in terms of the original coordinates.

The angular momenta $\wp_\theta$ and $\wp_\varphi$ are conserved quantities.

Now, $\wp_\theta$ has the value $I\Phi/\lambda$, where $\Phi$ is the value of the flux when $r = 0$ (or equivalently, when no electron is present).

Then $\dot{\theta}$ is given by

$$\dot{\theta} = \frac{\Phi}{\lambda} - \frac{e\lambda r^2\dot{\varphi}}{c^2\pi a^2 I} = \frac{\Phi}{\lambda} + \Delta\dot{\theta} \tag{31}$$

The cylinder phase shift is

$$\Delta\left(\frac{1}{\hbar}\int \wp_\theta d\theta\right) = \frac{1}{\hbar}\int \wp_\theta \Delta\dot{\theta} dt = -\frac{1}{\hbar}\int \Phi \frac{er^2}{c2\pi a^2} d\varphi = -\frac{eB}{\hbar c}\int \frac{r^2}{2} d\varphi \tag{32}$$

Over one revolution of the electron in its orbit, the phase shift is

$$-\frac{eB}{c\hbar} \cdot (\text{area of orbit}) = -\frac{e\Phi'}{c\hbar} \tag{33}$$

where $\Phi'$ is the flux encircled by the orbit. The conserved electron canonical angular momentum is

$$\wp_\varphi = \mu r^2\dot{\varphi} + \frac{e\lambda r^2\dot{\theta}}{c2\pi a^2} = \mu r^2\dot{\varphi} + \frac{er^2\Phi}{c2\pi a^2} - \frac{e^2\lambda^2 r^4}{c^2 4\pi^2 a^4 I} \tag{34}$$

In the limit $I \to \infty$, the term in $e^2/I$ in equation (34) may be dropped. The second term of equation (34) is the vector potential term. It gives a phase shift per cycle of

$$\frac{e}{\hbar c}\oint A_\varphi r d\varphi = \frac{e}{\hbar c}\oint \frac{r^2 B}{2} d\varphi = \frac{e}{\hbar c} B \cdot (\text{orbit area}) = \frac{e\Phi'}{\hbar c} \tag{35}$$

The phase shifts are again equal and opposite, as in the exterior region. It should be noted that for an AB experiment performed inside the solenoid, the phase difference would be given by equation (35) with $\Phi'$ being the flux between the two paths open to the electron.

It is enlightening to check the energy balance. The reader will recall that we have defined the flux so that $\Phi = \lambda\dot{\theta}$ when the electron is at the origin.

For electron orbits passing through $r = 0$, $\wp_\varphi$ vanishes. The change in the angular velocity of the cylinder is then

$$\Delta\dot{\theta} = -\frac{e\lambda r^2\dot{\varphi}}{c2\pi a^2 I} \tag{36}$$

The energy change of the rotating cylinder is

$$\Delta E_{\text{cyl}} = I\dot{\theta}\Delta\dot{\theta} \simeq -\frac{\Phi e r^2\dot{\varphi}}{c2\pi a^2} = -\frac{Be}{c}\frac{1}{2} r^2\dot{\varphi} = -\frac{e}{c}\boldsymbol{v}\cdot\mathbf{A} \tag{37}$$

This last term is the negative of the overlap magnetic field energy. The reader will recall that $\theta = \lambda\dot{\theta}$ when the electron is at the origin. When $r = 0$, $\boldsymbol{v}\cdot\mathbf{A}$ vanishes.

Equation (37) shows that the energy shift is in general time dependent, reflecting a lack of constancy in the angular velocity of the cylinder. As the electron moves in its orbit (which is fixed in space), kinetic energy of the cylinder is continually being exchanged with magnetic energy.

The average energy shift may be computed by means of a single semiclassical argument. The phase shift/cycle for the cylinder was found in equation (33) to be $-e\Phi'/c\hbar$.

A continuous shift in phase is just a shift in angular frequency. This frequency shift is the phase shift divided by the period of revolution of the electron.

Hence

$$\langle \Delta E_{\text{cyl}} \rangle = -\frac{\hbar e\Phi'}{c\hbar} \frac{eB}{2\pi\mu c} = -\frac{e^2B^2 \cdot (\text{area of orbit})}{2\pi\mu c^2} \tag{38}$$

From equation (37),

$$\langle \Delta E_{\text{cyl}} \rangle = -\frac{Be}{c} \int \frac{1}{2} r^2 \dot{\phi} dt \cdot \frac{eB}{2\pi\mu c} \tag{39}$$

The integral in equation (39) is the area of the electron orbit. Equations (38) and (39) yield

$$\langle \Delta E_{\text{cyl}} \rangle = -\frac{1}{2} \mu v^2 \tag{40}$$

The purpose of this discussion has been to show clearly the difference in behavior of the cylinder when the electron is in the magnetic field and when it is in the exterior region. When the electron is in the magnetic field, the cylinder undergoes a continuous phase shift, i.e., an energy shift. This energy continually oscillates between kinetic energy and energy of the magnetic field. As in the AB effect, the kinetic energy of the electron is unchanged.

An electron in the exterior region travels in a straight line. The cylinder gets only a one-time phase shift—as does the wave function of the electron.

If now one could muster the energy to solve the Schrödinger equation for this composite system, electron plus cylinder in all of space, one would find a correlation between the coordinates of the electron and the kinetic angular momentum of the cylinder. Even the qualitative behavior of the cylinder is different in the two cases.

We have already seen that in the AB problem, the price that one pays for making an external field approximation is that the single-valuedness of the electron wave function must be sacrificed. If one were to go further, using an external field approximation over all space, one would omit a degree of freedom that is correlated with the motion of the electron. This comes at a still higher price. The solution is good in the interior region. However, the behavior of the omitted cyclinder is quite different as the electron approaches the boundary of the cylinder from the two sides of the cylinder wall. It is clear that the wave function obtained under such an omission cannot possibly be continuous at the cylinder wall. This lack of continuity was discussed by the author (Henneberger, 1984) 13 years ago. The earlier derivation was based upon continuity of the transport of tangential probability current across the boundary of a solenoid.

## 4. CONCLUSION

In summary, it has been shown that:

1. In order to discuss the AB effect in a system that truly conserves energy, one must represent the solenoid as having an internal degree of freedom. Here it has been represented as a rotating cylinder. Solutions of the complete AB problem (electron plus cylinder) have been given. The earlier result of Henneberger and Opatrny has been demonstrated for the composite system by means of a transformation to normal coordinates. The method of diagonalizing a quadratic form was applied long ago to light scattering problems by van Kampen (1951) and Steinwedel (1955). The method goes back to the early days of quantum theory, and the transformation of Kramers (1938).

2. The model has been extended to the case of electrons confined to the interior of the cylinder by the magnetic field. Such states form a complete set. Since the electron cannot sense the presence of the cylinder wall until it reaches it, any state, bound or not, may be expanded in terms of states for which the magnetic field extends to infinity.

3. The rotating cylinder exhibits different behaviors when an electron is on different sides of the cylinder wall. The external field approximation forces a discontinuity in the wave function at the cylinder wall. The AB effect has historically aroused so much interest because it has been (and continues to be) ill understood. The standard papers on the AB effect have the common problem of having too few degrees of freedom. The torque on the solenoid is typically assumed to vanish. It does not. The electron kinetic energy is constant, but the magnetic energy (which is typically ignored) is not.

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